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Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Theorem

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

- 1 Find the range of the values of x which satisfy both inequalities
 $0 < x^2 - 4x$ and $x^2 - 4x \leq 3x + 10$. [4]
- 2 Solve
- (i) $\frac{3^{2-x}}{9^x} = \frac{1}{\sqrt{27^x}}$, [2]
- (ii) $3e^x - e = 2e^{2-x}$. [3]
- 3 (i) Find the coefficient of the term in x in the expansion of $\left(x^2 - \frac{1}{2x^3}\right)^8$. [3]
- (ii) The coefficient of x^2 in the expansion $(5-3x)(1+5x)^n$ is 1785. Find the value of n . [4]
- 4 The gradient to a curve is given by $\frac{dy}{dx} = (kx+3)^2$, where k is a non-zero constant. The equation of the tangent to the curve at the point $(1, 2)$ is $9x - y - 5 = 0$. Find the
- (i) value of k , [2]
- (ii) equation of the curve. [2]
- 5 Sketch the graph of $y = -|x+1| + 2$ for $-4 \leq x \leq 2$. [3]
- (i) State the range of values of p for which the equation $-|x+1| = p - 2$ has at least 1 solutions for $-4 \leq x \leq 2$. [1]
- (ii) Using your graph, state the number of solutions for $-|x+1| + 2 = x + 3$. [1]
- 6 (i) Find the exact value of x in the equation $\sqrt{12}x + 5 = \sqrt{7}x + 19$. [4]
- (ii) A cuboid with a square base of length $\sqrt{3} + 1$ cm, has a volume of $(5\sqrt{2})^2 - 8\sqrt{3}$ cm³. Find the height of the cuboid in the form $a + b\sqrt{3}$. [4]

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- 7 A curve has the equation $y = xe^{4x}$.

(i) Find $\frac{dy}{dx}$. [2]

(ii) Hence show that $\int_0^{\ln 2} 4xe^{4x} dx = 16\ln 2 - 3\frac{3}{4}$. [4]

(iii) Find the range of values of x for which the function $y = xe^{4x}$ is decreasing. [2]

- 8 AB is a chord of the circle $x^2 + y^2 - 8x - 2y - 3 = 0$ and $M\left(\frac{4}{5}, 2\frac{2}{5}\right)$ is the midpoint of chord AB . Find the

(i) radius and the coordinates of the centre of the circle, [2]

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If P is a variable point on the circle, find the

(iii) maximum area of triangle ABP . [4]

- 9 The function f is defined, for $0 \leq x \leq 2\pi$, by $f(x) = 2\cos ax + b$, where a and b are integers. The minimum value of f is -1 and the period of f is $\frac{4\pi}{3}$.

(i) State the amplitude of f . [1]

(ii) State the values of a and of b . [1]

(iii) Using the values of a and b found in part (ii),

(a) solve $f(x) = 0$ for $0 \leq x \leq 2\pi$, leaving your answers in terms of π , [4]

(b) sketch the graph of $f(x) = 2\cos ax + b$ for $0 \leq x \leq 2\pi$. [3]

- 10 A particle moves in a straight line such that t seconds after leaving a fixed point O , the velocity v m/s, is given by $v = 3t^2 - t - 10$. Find the

(i) initial acceleration of the particle, [2]

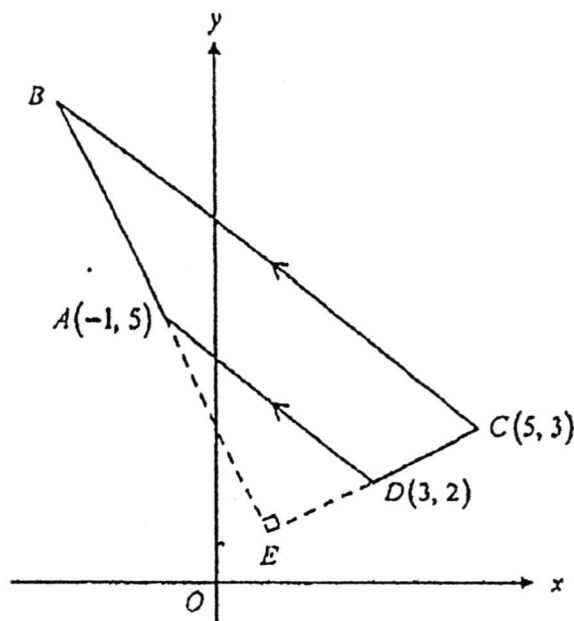
(ii) minimum velocity of the particle, [2]

(iii) total distance travelled by the particle in the first 3 seconds, [4]

(iv) average speed of the particle during the first 3 seconds. [2]

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The diagram shows the trapezium $ABCD$ in which BC is parallel to AD while BA produced is perpendicular to CD produce at point E . The point A is $(-1, 5)$, C is $(5, 3)$ and D is $(3, 2)$.



- (i) Show that the coordinates of B are $(-3, 9)$. [6]
- (ii) Find the area of trapezium $ABCD$. [2]
- (iii) Given that $\frac{\text{area of } \triangle AED}{\text{area of } \triangle BEC} = \frac{1}{4}$, find the coordinates of E . [3]

End of Paper

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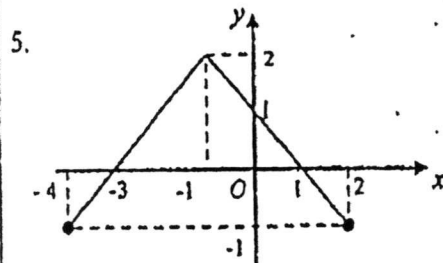
Answer Key

1. $-1.22 \leq x < 0$ or $4 < x \leq 8.22$

2(i) $1\frac{1}{3}$ 2(ii) $x=1$

3(i) Coefficient of $x = -7$ 3(ii) $n=6$

4(i) $k = -6$ 4(ii) $y = \frac{1}{2} - \frac{3(1-2x)^3}{2}$



5(i) $-1 \leq p \leq 2$

5(ii) There are infinite number of solutions.

6(i) $x = \frac{2\sqrt{7}}{3}$ 6(ii) height = $62 - 33\sqrt{3}$

7(i) $\frac{dy}{dx} = e^{4x}(4x+1)$ 7(iii) $x < -\frac{1}{4}$

8(i) centre of circle is $(4, 1)$

(ii) radius = 4.47 units

(iii) Max area = 22.2 units²9(i) Amplitude = 2, (ii) $a=1.5$, $b=1$

(iii) (a) $x = \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}$

10(i) -1 m/s^2

(ii) min velocity = $-10\frac{1}{12} \text{ m/s}$

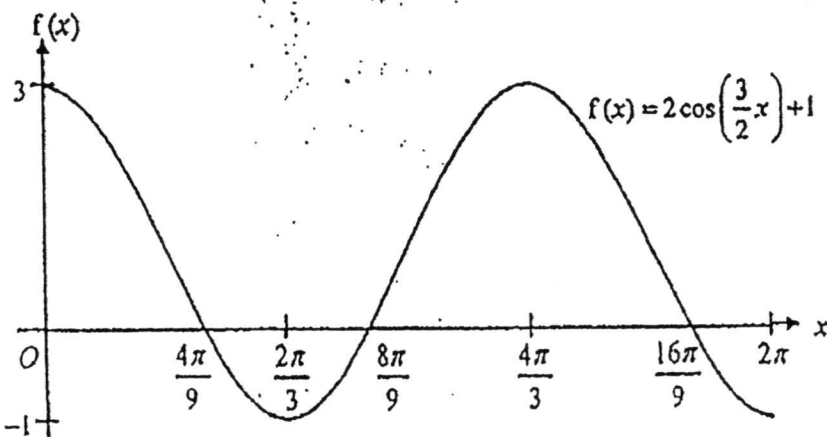
(iii) Total Distance = 20.5 m

(iv) Ave Speed = $6\frac{5}{6}$ or 6.83 m/s

11(ii) Area = 15 units²

(iii) $E(1, 1)$

9(ii)(b)



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where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

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 $0 < x^2 - 4x$ and $x^2 - 4x \leq 3x + 10$. [4]

$$0 < x^2 - 4x \quad \text{and} \quad x^2 - 4x \leq 3x + 10$$

$$x^2 - 4x > 0 \quad x^2 - 7x - 10 \leq 0$$

$$x(x-4) > 0 \quad \text{for } x^2 - 7x - 10 = 0$$

$$x < 0 \text{ or } x > 4 \quad x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-10)}}{2(1)}$$

$$= -1.22 \text{ or } 8.22$$

$$\therefore x^2 - 7x - 10 \leq 0$$

$$-1.22 \leq x \leq 8.22$$

Hence the solution is $-1.22 \leq x < 0$ or $4 < x \leq 8.22$

- 2 Solve

$$(i) \quad \frac{3^{2-x}}{9^x} = \frac{1}{\sqrt{27^x}}, \quad [2]$$

$$\frac{3^{2-x}}{9^x} = \frac{1}{\sqrt{27^x}}$$

$$\frac{3^{2-x}}{3^{2x}} = \frac{1}{3^{\frac{x}{2}}}$$

$$2 - x - 2x = -\frac{3x}{2}$$

$$2 = \frac{3}{2}x$$

$$x = \frac{4}{3} = 1\frac{1}{3}$$

$$(ii) \quad 3e^x - e = 2e^{2-x}. \quad [3]$$

$$3e^x - e = 2e^{2-x}$$

$$3e^x - e = \frac{2e^2}{e^x}$$

$$3(e^x)^2 - e \cdot e^x - 2e^2 = 0$$

$$(e^x - e)(3e^x + 2e) = 0$$

$$e^x = e \quad \text{or} \quad 3e^x = -2e$$

$$x = 1 \quad (\text{NA})$$

- 3 (i) Find the coefficient of the term in x in the expansion of $\left(x^2 - \frac{1}{2x^3}\right)^8$. [3]

$$\text{For } \left(x^2 - \frac{1}{2x^3}\right)^8,$$

$$T_{r+1} = \binom{8}{r} (x^2)^{8-r} \left(-\frac{1}{2x^3}\right)^r$$

$$= \binom{8}{r} \left(\frac{-1}{2}\right)^r x^{16-2r} x^{-3r}$$

$$= \binom{8}{r} \left(\frac{-1}{2}\right)^r x^{16-5r}$$

$$\text{For term in } x, \quad 16-5r=1$$

$$5r=15$$

$$r=3$$

$$\text{Coefficient of } x = \binom{8}{3} \left(\frac{-1}{2}\right)^3$$

$$=-7$$

- (ii) The coefficient of x^2 in the expansion $(5-3x)(1+5x)^n$ is 1785. Find the value of n . [4]

$$(5-3x)(1+5x)^n$$

$$= (5-3x) \left(1 + \binom{n}{1}(5x) + \binom{n}{2}(5x)^2 + \dots\right)$$

$$= (5-3x) \left(1 + 5nx + \frac{n(n-1)}{2} \times 25x^2 + \dots\right)$$

coefficient of x^2 in the above expansion = 1785

$$125 \times \frac{n(n-1)}{2} - 3(5n) = 1785$$

$$125n(n-1) - 30n = 3570$$

$$125n^2 - 125n - 30n - 3570 = 0$$

$$125n^2 - 155n - 3570 = 0$$

$$25n^2 - 31n - 714 = 0$$

$$(n-6)(25n+119) = 0$$

$$n-6=0 \quad \text{or} \quad 25n+119=0$$

$$n=6 \quad \text{or} \quad n = -\frac{119}{25} \text{ (N.A.)}$$

- 4 The gradient to a curve is given by $\frac{dy}{dx} = (kx+3)^2$, where k is a non-zero constant. The equation of the tangent to the curve at the point $(1, 2)$ is $9x - y - 5 = 0$. Find the

(i) value of k ,

[2]

$$9x - y - 5 = 0$$

$$y = 9x - 5$$

$$\text{Gradient of tangent} = 9$$

$$\text{At } (1, 2), \frac{dy}{dx} = 9$$

$$(k+3)^2 = 9$$

$$k+3 = 3 \text{ or } k+3 = -3$$

$$k = 0 \text{ (N.A.) or } k = -6$$

(ii) equation of the curve.

[2]

Equation of curve is,

$$y = \int (-6x+3)^2 dx$$

$$= \frac{(-6x+3)^3}{3(-6)} + c$$

$$= \frac{(3-6x)^3}{-18} + c$$

$$\text{At } (1, 2),$$

$$2 = \frac{(3-6)^3}{-18} + c$$

$$c = \frac{1}{2}$$

$\therefore$ equation of curve is,

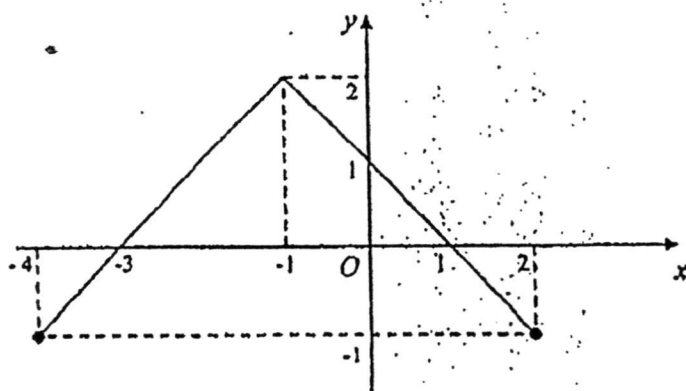
$$y = \frac{(3-6x)^3}{-18} + \frac{1}{2}$$

$$y = \frac{1}{2} - \frac{3(1-2x)^3}{2}$$

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- 5 Sketch the graph of $y = -|x+1| + 2$ for $-4 \leq x \leq 2$.

[3]



- (i) State the range of values of p for which the equation $-|x+1| = p-2$ has at least 1 solutions for $-4 \leq x \leq 2$.

[1]

$$-1 \leq p \leq 2$$

- (ii) Using your graph, state the number of solutions for $-|x+1| + 2 = x+3$

[1]

There are infinite number of solutions.

- 6 (i) Find the exact value of x in the equation $\sqrt{112}x + 5 = \sqrt{7}x + 19$.

[4]

$$\sqrt{112}x + 5 = \sqrt{7}x + 19$$

$$4\sqrt{7}x - \sqrt{7}x = 14$$

$$3\sqrt{7}x = 14$$

$$x = \frac{14}{3\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{14\sqrt{7}}{21}$$

$$= \frac{2\sqrt{7}}{3}$$

$$\sqrt{112}x + 5 = \sqrt{7}x + 19$$

$$\sqrt{112}x - \sqrt{7}x = 14$$

$$(\sqrt{112} - \sqrt{7})x = 14$$

$$x = \frac{14}{\sqrt{112} - \sqrt{7}} \times \frac{\sqrt{112} + \sqrt{7}}{\sqrt{112} + \sqrt{7}}$$

$$= \frac{14 \cdot 4\sqrt{7} + 14\sqrt{7}}{112 - 7}$$

$$= \frac{56\sqrt{7} + 14\sqrt{7}}{105}$$

$$= \frac{70\sqrt{7}}{105}$$

$$= \frac{2\sqrt{7}}{3}$$

- (ii) A cuboid with a square base of length $\sqrt{3} + 1$ cm, has a volume of $(5\sqrt{2})^2 - 8\sqrt{3}$ cm³. Find the height of the cuboid in the form $a + b\sqrt{3}$.

[4]

$$\text{Height} = \frac{(5\sqrt{2})^2 - 8\sqrt{3}}{(\sqrt{3} + 1)^2}$$

$$= \frac{25(2) - 8\sqrt{3}}{4 + 2\sqrt{3}} \times \frac{4 - 2\sqrt{3}}{4 - 2\sqrt{3}}$$

$$= \frac{200 - 100\sqrt{3} - 32\sqrt{3} + 48}{16 - 12}$$

$$= \frac{248 - 132\sqrt{3}}{4}$$

$$= 62 - 33\sqrt{3}$$

- 7 A curve has the equation $y = xe^{4x}$.

(i) Find $\frac{dy}{dx}$.

[2]

$$y = xe^{4x}$$

$$\frac{dy}{dx} = x4e^{4x} + e^{4x} (1)$$

$$= e^{4x} (4x + 1)$$

(ii) Hence show that $\int_0^{\ln 2} 4xe^{4x} dx = 16\ln 2 - 3\frac{3}{4}$.

[4]

$$\int_0^{\ln 2} e^{4x} (4x + 1) dx = [xe^{4x}]_0^{\ln 2}$$

$$\int_0^{\ln 2} 4xe^{4x} + e^{4x} dx = \ln 2 \times e^{4\ln 2} - 0$$

$$\int_0^{\ln 2} 4xe^{4x} dx + \int_0^{\ln 2} e^{4x} dx = \ln 2 \times e^{\ln 16}$$

$$\int_0^{\ln 2} 4xe^{4x} dx = \ln 2 \times 16 - \int_0^{\ln 2} e^{4x} dx$$

$$= 16\ln 2 - \int_0^{\ln 2} e^{4x} dx$$

$$= 16\ln 2 - \left[\frac{e^{4x}}{4} \right]_0^{\ln 2}$$

$$= 16\ln 2 - \frac{1}{4} (e^{4\ln 2} - e^0)$$

$$= 16\ln 2 - \frac{1}{4} (e^{\ln 16} - 1)$$

$$= 16\ln 2 - \frac{1}{4} (16 - 1)$$

$$= 16\ln 2 - 3\frac{3}{4}$$

- (iii) Find the range of values of x for which the function $y = xe^{4x}$ is decreasing.

[2]

For y to be decreasing,

$$\frac{dy}{dx} < 0$$

$$e^{4x} (4x + 1) < 0$$

Since $e^{4x} > 0$ for all values of x ,

$$\text{then } 4x + 1 < 0$$

$$x < -\frac{1}{4}$$

- 8 AB is a chord of the circle $x^2 + y^2 - 8x - 2y - 3 = 0$ and $M\left(\frac{4}{5}, 2\frac{2}{5}\right)$ is the midpoint of chord AB . Find the

(i) radius and the coordinates of the centre of the circle,

[2]

$$x^2 + y^2 - 8x - 2y - 3 = 0$$

$$x^2 + y^2 + 2(-4)x + 2(-1)y + (-3) = 0$$

$$\therefore g = -4, f = -1, c = -3$$

Hence centre of circle is $(4, 1)$

$$\text{radius of circle is } \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(-4)^2 + (-1)^2 - (-3)}$$

$$= \sqrt{20}$$

$$= 4.47 \text{ units (3 sf)}$$

(ii) equation of chord AB .

[3]

Let the centre of circle be C .

$$\begin{aligned} \therefore \text{gradient of } CM &= \frac{2\frac{2}{5} - 1}{\frac{4}{5} - 4} \\ &= \frac{-\frac{3}{5}}{-\frac{16}{5}} \\ &= \frac{3}{16} \end{aligned}$$

$$\therefore \text{gradient of chord } AB = \frac{16}{3}$$

Hence equation of chord AB is,

$$y - 2\frac{2}{5} = \frac{16}{3}\left(x - \frac{4}{5}\right)$$

$$= \frac{16}{3}x - \frac{64}{15}$$

$$\therefore y = \frac{16}{3}x + \frac{4}{15}$$

If P is a variable point on the circle, find the

(iii) maximum area of triangle ABP . [4]

$$\begin{aligned} \text{Length of } CM &= \sqrt{\left(4 - \frac{4}{5}\right)^2 + \left(1 - 2\frac{2}{5}\right)^2} \\ &= \sqrt{12\frac{1}{5}} \end{aligned}$$

$$\begin{aligned} \text{Length of } BM &= \sqrt{BC^2 - CM^2} \\ &= \sqrt{20 - 12\frac{1}{5}} \\ &= \sqrt{7\frac{4}{5}} \end{aligned}$$

$$\text{Length of chord } AB = 2 \times BM$$

$$= 2 \times \sqrt{7\frac{4}{5}}$$

Area of $\triangle ABP$ is maximum when P, C & M are collinear and PM is $\perp AB$.

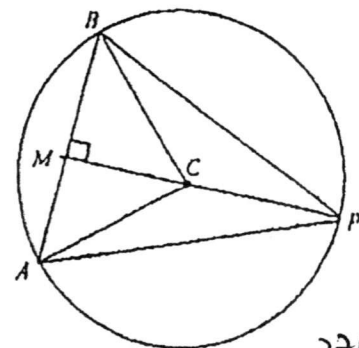
$\therefore$ maximum area of $\triangle ABP$

$$= \frac{1}{2} \times AB \times PM$$

$$= \frac{1}{2} \times AB \times (CM + CP)$$

$$= \frac{1}{2} \times \left(2 \times \sqrt{7\frac{4}{5}}\right) \times \left(\sqrt{12\frac{1}{5}} + \sqrt{20}\right)$$

$$= 22.2 \text{ units}^2 \text{ (3 sf)}$$



- 9 The function f is defined, for $0 \leq x \leq 2\pi$, by $f(x) = 2\cos ax + b$, where a and b are integers. The minimum value of f is -1 and the period of f is $\frac{4\pi}{3}$.

(i) State the amplitude of f . [1]

Amplitude = 2

(ii) State the values of a and of b . [1]

$$a = 2\pi \div \frac{4\pi}{3} = 1.5$$

$$b = 1$$

(iii) Using the values of a and b found in part (ii),
(a) solve $f(x) = 0$ for $0 \leq x \leq 2\pi$, leaving your answers in terms of π . [4]

$$2\cos\left(\frac{3}{2}x\right) + 1 = 0$$

$$\cos\left(\frac{3}{2}x\right) = -\frac{1}{2}$$

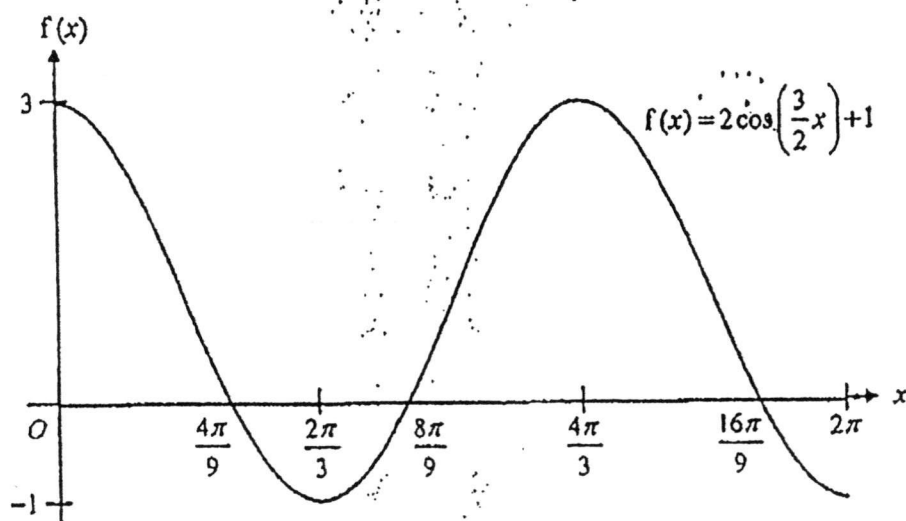
$$\alpha = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\frac{3}{2}x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 2\pi + \left(\pi - \frac{\pi}{3}\right)$$

$$\frac{3}{2}x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

$$x = \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}$$

(b) sketch the graph of $f(x) = 2\cos ax + b$ for $0 \leq x \leq 2\pi$. [3]



- 10 A particle moves in a straight line such that t seconds after leaving a fixed point O , the velocity v m/s, is given by $v = 3t^2 - t - 10$. Find the

- (i) initial acceleration of the particle, [2]

$$v = 3t^2 - t - 10$$

$$a = \frac{dv}{dt}$$

$$= 6t - 1$$

$$\text{Initial acceleration} = 6(0) - 1$$

$$= -1 \text{ m/s}^2$$

- (ii) minimum velocity of the particle, [2]

Minimum velocity of the particle occurs when $a = 0$

$$6t - 1 = 0$$

$$t = \frac{1}{6}$$

$\therefore$ minimum velocity of the particle,

$$= 3\left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right) - 10 = -10\frac{1}{12} \text{ m/s}$$

- (iii) total distance travelled by the particle in the first 3 seconds, [4]

$$s = \int 3t^2 - t - 10 \, dt$$

$$= t^3 - \frac{1}{2}t^2 - 10t + c$$

$$\text{At } t = 0, s = 0, \therefore c = 0$$

$$\text{Hence, } s = t^3 - \frac{1}{2}t^2 - 10t$$

$$\text{When } v = 0,$$

$$3t^2 - t - 10 = 0$$

$$(3t + 5)(t - 2) = 0$$

$$3t + 5 = 0 \text{ or } t - 2 = 0$$

$$t = \frac{-5}{3} \text{ (N.A.) } \quad t = 2$$

$$\text{At } t = 2,$$

$$s = 2^3 - \frac{1}{2}(2)^2 - 10(2) = -14$$

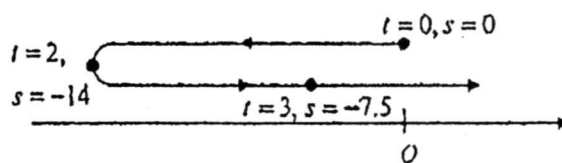
$$\text{At } t = 3,$$

$$s = 3^3 - \frac{1}{2}(3)^2 - 10(3) = -7\frac{1}{2}$$

$\therefore$ total distance travelled in the first 3 seconds

$$= 14 + (14 - 7.5)$$

$$= 20.5 \text{ m}$$



- (iv) the average speed of the particle during the first 3 seconds. [2]

Average speed of the particle during the first 3 seconds

$$= \frac{\text{total distance}}{\text{total time}} = \frac{20.5}{3}$$

$$= 6\frac{5}{6} \text{ or } 6.83 \text{ m/s}$$

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The diagram shows the trapezium $ABCD$ in which BC is parallel to AD while BA produced is perpendicular to CD produce at point E . The point A is $(-1, 5)$, C is $(5, 3)$ and D is $(3, 2)$.

- (i) Show that the coordinates of B are $(-3, 9)$.

Gradient of BC = Gradient of AD

$$\begin{aligned} &= \frac{5-2}{-1-3} \\ &= -\frac{3}{4} \end{aligned}$$

Sub $(5, 3)$ into $y = -\frac{3}{4}x + c$, $3 = -\frac{3}{4}(5) + c$

$$c = 6\frac{3}{4}$$

Equation of BC is $y = -\frac{3}{4}x + 6\frac{3}{4}$

$$\begin{aligned} \text{Gradient of } CD &= \frac{3-2}{5-3} \\ &= \frac{1}{2} \end{aligned}$$

Gradient of BA = -2

Sub $(-1, 5)$ into $y = -2x + d$, $5 = -2(-1) + d$

$$d = 3$$

Equation of BA is $y = -2x + 3$

$$-\frac{3}{4}x + 6\frac{3}{4} = -2x + 3$$

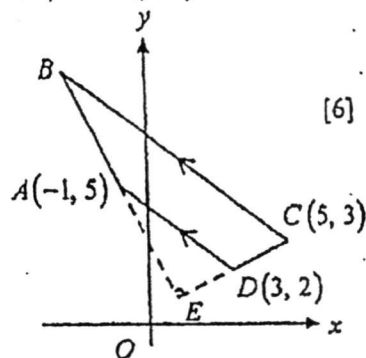
$$-3x + 27 = -8x + 12$$

$$5x = -15$$

$$x = -3$$

when $x = -3$, $y = -2(-3) + 3$
 $= 9$

$$\therefore B(-3, 9)$$

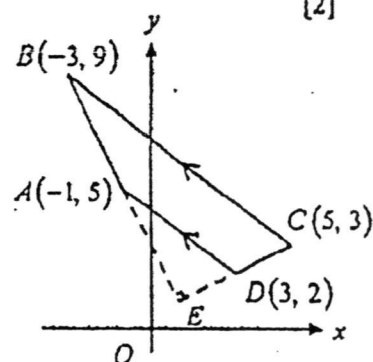


- 11 (ii) Find the area of trapezium
- $ABCD$
- .

[2]

Area of trapezium $ABCD$

$$\begin{aligned}
 &= \frac{1}{2} \begin{vmatrix} 5 & -3 & -1 & 3 & 5 \\ 3 & 9 & 5 & 2 & 3 \end{vmatrix} \\
 &= \frac{1}{2} [(45 - 15 - 2 + 9) - (-9 - 9 + 15 + 10)] \\
 &= \frac{1}{2} (37 - 7) \\
 &= 15 \text{ units}^2
 \end{aligned}$$



- (iii) Given that
- $\frac{\text{area of } \triangle AED}{\text{area of } \triangle BEC} = \frac{1}{4}$
- , find the coordinates of
- E
- .

[3]

Since $\triangle MED$ and $\triangle BEC$ are similar,

$$\frac{ED}{EC} = \frac{EA}{EB} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$\therefore D$ and E are the midpoints
of EC and EB respectively.

Let $E(m, n)$,

$$\left(\frac{5+m}{2}, \frac{3+n}{2} \right) = (3, 2)$$

$$\frac{5+m}{2} = 3 \quad \frac{3+n}{2} = 2$$

$$m = 1 \quad n = 1$$

$$\therefore E(1, 1) \quad \leftarrow A1$$

or

$$\left(\frac{-3+m}{2}, \frac{9+n}{2} \right) = (-1, 5)$$

$$\text{Sub } (3, 2) \text{ into } y = \frac{1}{2}x + f,$$

$$2 = \frac{1}{2}(3) + f$$

$$f = \frac{1}{2}$$

$$\text{Equation of } CD \text{ is } y = \frac{1}{2}x + \frac{1}{2}$$

$$\text{Equation of } BA \text{ is } y = -2x + 3$$

$$\frac{1}{2}x + \frac{1}{2} = -2x + 3$$

$$x + 1 = -4x + 6$$

$$5x = 5$$

$$x = 1$$

$$\text{when } x = 1, \quad y = -2(1) + 3 = 1$$

$$\therefore E(1, 1)$$

End of Paper

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Theorem

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \dots + \binom{n}{r} a^{n-r}b^r + \dots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

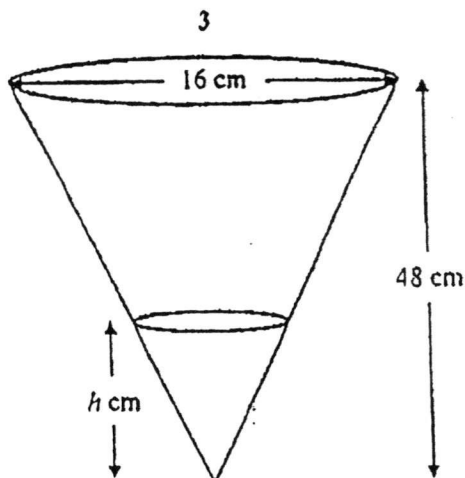
Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

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Water from a tank in the shape of an inverted cone flows out at the rate of $5 \text{ cm}^3/\text{min}$. The height of the cone is 48 cm and the base diameter is 16 cm. After t minutes the water level is h cm.

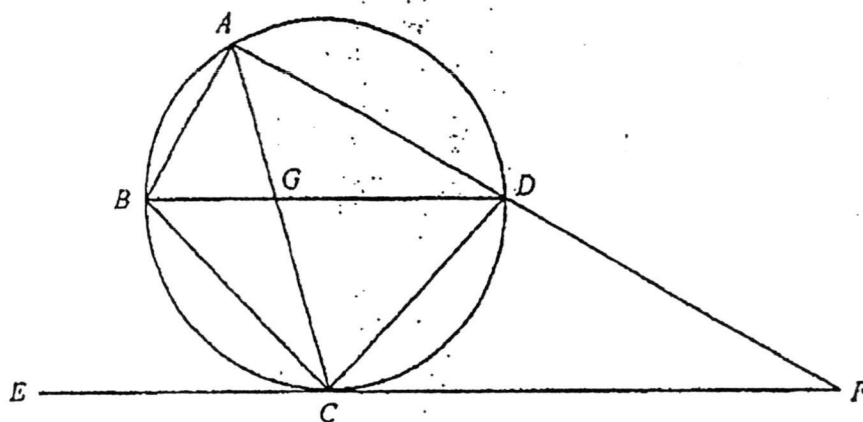
- (i) Show that the volume of water in the tank, $V \text{ cm}^3$, at time t is given by $V = \frac{\pi h^3}{108}$. [2]
- (ii) Find the rate of change of the water level when $h = 6$. [3]
- (iii) State, with a reason, whether this rate will increase or decrease as t increases. [1]

- 2 The displacement, y mm, of a mass fixed on a vertical spring can be described by the simple harmonic motion equation, $y = A \sin(\omega t)$, where A and ω are constants and t is the time in seconds after the mass is displaced from its equilibrium position, 0 mm.

Given that the maximum displacement of the mass is 20 mm and that the mass first returns to its equilibrium position after 0.25 seconds.

- (i) State the positive value of A . [1]
- (ii) Show that the value of ω is 4π radians per second. [2]
- (iii) Find the exact value of t when the mass first reach a position 10 mm below its equilibrium position. [3]

- 3 (i) Given that $f(x) = 2x^3 + ax^2 + bx - 30$ has a factor $(x+3)$ and leaves a remainder of -28 when divided by $(x-1)$. Find the values of a and of b and solve $f(x) = 0$. [6]
- (ii) Hence solve $2(y+1)^3 + a(y+1)^2 + by + b - 30 = 0$. [2]



The diagram shows points A, B, C and D lying on a circle. The chords BD and AC intersect at G . EF is a tangent to the circle at C . AD is produced meet the tangent at F and $\angle ABC = \angle BGC$.

Prove that

(i) BD is parallel to EF , [2]

(ii) triangle CFD and triangle AFC are similar, [2]

(iii) $FC^2 - FD^2 = FD \times DA$. [3]

5 (i) Express $\frac{3x^2 + 10x}{(x+2)(x^2-4)}$ in partial fractions. [5]

(ii) Using your answer from (i), find $\int \frac{3x^2 + 10x}{(x+2)(x^2-4)} dx$ and hence show that

$$\int_3^4 \frac{3x^2 + 10x}{(x+2)(x^2-4)} dx = \ln\left(\frac{24}{5}\right) + \frac{1}{15}. \quad [4]$$

6 (a) The quadratic equation $3x^2 - 2x + 4 = 0$ has roots $3\alpha + \beta$ and $\alpha + 3\beta$.

(i) Show that the values of $\alpha + \beta = \frac{1}{6}$ and $\alpha\beta = \frac{5}{16}$. [4]

(ii) If the roots of the equation $gx^2 - hx - 1 = 0$ where g and h are constants, are α and β , find the value of g and of h . [2]

(b) Find the range of values of k for which $(k+3)x^2 + kx + 1$ is always positive for all real values of x . [4]

7 Answer the whole of this question on a sheet of graph paper.

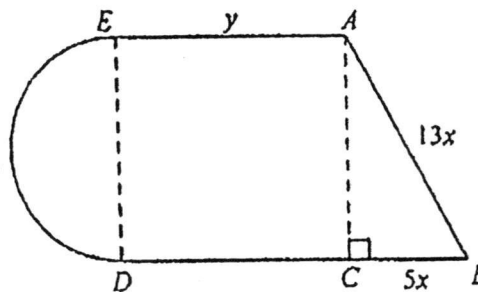
The table shows experimental values of two variables, x and y .

x	0.4	0.6	0.8	1.0	1.2
y	2.22	2.13	1.97	1.73	1.37

It is known that x and y are related by the equation $y^2 = (ax+1)x - b$, where a and b are constants.

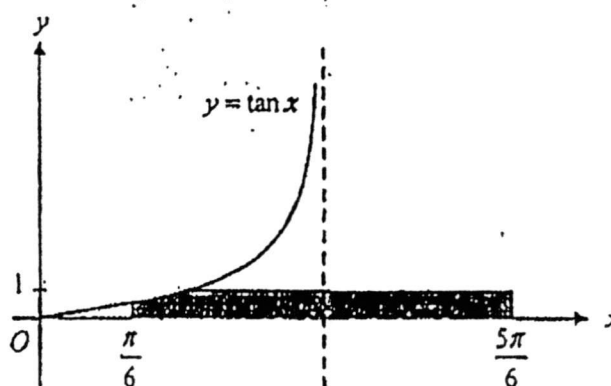
- (i) On graph paper, plot $(y^2 - x)$ against x^2 , using a scale of 2 cm to represent 0.2 unit on the x^2 -axis and 4 cm to represent 1 unit on the $(y^2 - x)$ -axis. Draw a straight line graph to represent the equation $y^2 = (ax+1)x - b$. [3]
- (ii) Use your graph to estimate the value of a and of b . [4]
- (iii) By drawing a suitable straight line on your graph, solve the equation $(a-2) = \frac{1+b}{x^2}$. [2]

- 8 A piece of wire 160 cm long is bent to form the shape shown in the figure. This shape consists of a semi-circular arc whose diameter is given by the length DE , and a right-angled triangle ABC on the opposite ends of a rectangle of length y cm. The length of BC and AB are $5x$ cm and $13x$ cm respectively.



- (i) Express y in terms of x . [2]
- (ii) Show that the area enclosed, $A \text{ cm}^2$, is given by $A = 960x - 6(3\pi + 13)x^2$. [2]
- (iii) Determine the value of x for which A has a stationary value. [3]
- (iv) Find the stationary value of A and determine if it is a maximum or a minimum value. [3]

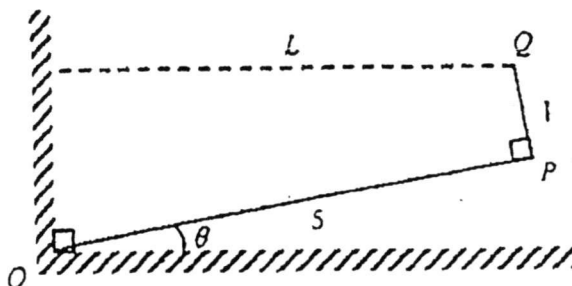
- 9 (a) (i) Prove that $\cos A = \frac{\cos 2A}{\cos A} + \tan A \sin A$. [3]
- (ii) Solve, for $0^\circ \leq A \leq 360^\circ$, $\cos A - \tan A \sin A = -1$. [5]
- (b) Given $\cos \theta = -\frac{4}{5}$ and θ is in the third quadrant. Without using a calculator, find the value of $\cos \frac{\theta}{2}$. [3]
- 10 (a) Solve the equation $\log_2 \frac{1}{2} = \log_2 x - \log_4 (9x-2)$. [3]
- (b) Given that $\log_2 (x+3) - (\log_2 y)(\log_8 2) = 2$, express y in terms of x . [3]
- (c) (i) Differentiate $\ln \cos x$. [1]
- (ii) State the principal value of $\tan^{-1} 1$, giving your answer as a multiple of π . [1]



The diagram shows part of the graph $y = \tan x$. The shaded region is bounded by the curve, the x -axis, lines $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$ and $y = 1$.

- (iii) Using your results from (i) and (ii), or otherwise, find the area of the shaded region. [4]

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A L-shaped structure, OPQ , can be rotated about O . OP and PQ measures 5 m and 1 m respectively. OP makes an acute angle, θ , with the ground. Given that L m is the shortest distance from Q to the wall,

- (i) show that $L = 5 \cos \theta - \sin \theta$, [2]
- (ii) express L in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, [4]
- (iii) state the minimum value of L and find the corresponding value of θ , [3]
- (iv) find the value of θ when $L = 3$, [2]
- (v) explain why the maximum value of L is not R . [1]

End of Paper

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Answer Key

1(ii) $\frac{dh}{dt} = -1.59 \text{ cm/min}$

(iii) As t increases, h decreases.

Since $\frac{dh}{dt} = \frac{-180}{\pi h^2}$, $\frac{dh}{dt}$ is inversely proportional to h^2 ,
hence rate of change of water level increases when h decreases.

Or

$$\frac{dh}{dt} = \frac{-180}{\pi h^2} \therefore \frac{d^2h}{dt^2} = \frac{360}{\pi h^3}$$

— Since $h^3 > 0$ for all positive h , then $\frac{d^2h}{dt^2} > 0$.

Hence $\frac{dh}{dt}$ is an increasing function.

2(i) $A = 20$ (iii) $t = \frac{7}{24}$

3(i) $a = 7$, $b = -7$
 $x = -3$ or $x = 2$ or $x = -2.5$

(ii) $y = -4$ or $y = 1$ or $y = -3.5$

4 Plane Geometry

5(i) $\frac{2}{x-2} + \frac{1}{x+2} + \frac{2}{(x+2)^2}$

6a(ii) $g = -3\frac{1}{5}$ $h = -\frac{8}{15}$

6b $-2 < k < 6$

7(ii) $a = -3.00$ $b = -5$

7(iii) Draw $y^2 - x = 2x^2 + 1$
 $x = \pm 0.894$

8(i) $y = 80 - 3(\pi + 3)x$

8(iii) $x = 3.57$

8(iv) Stationary value of $A = 1710$
 A is maximum

9a(ii) $A = 60^\circ, 180^\circ, 300^\circ$

9(b) $\cos \frac{\theta}{2} = -\frac{\sqrt{10}}{10}$

10(a) $x = \frac{1}{4}$ or $x = 2$

(b) $y = \frac{(x+3)^3}{64}$

(c) (i) $-\tan x$

(ii) Principal value of $\tan^{-1} 1 = \frac{\pi}{4}$

(iii) Area = 2.04 units²

11(ii) $L = 5.10 \cos(\theta + 11.3^\circ)$

(iii) Min $L = 0$ when $\theta = 78.7^\circ$

(iv) $\theta = 42.7^\circ$

(v) If $L = R$ then $\theta < 0^\circ$.

Since $0^\circ \leq \theta < 90^\circ$, $\therefore$ maximum $L \neq R$.

[Since $\theta \geq 0^\circ$, maximum L occurs when $\theta = 0^\circ$, maximum $L = 5$.]

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Theorem

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

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Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

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Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

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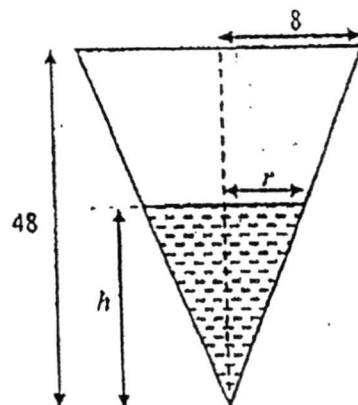
- 1 Water from a tank in the shape of an inverted cone flows out at the rate of $5 \text{ cm}^3/\text{min}$. The height of the cone is 48 cm and the base diameter is 16 cm . After t minutes the water level is $h \text{ cm}$.

- (i) Show that the volume of water in the tank, $V \text{ cm}^3$, at time t is given by $V = \frac{\pi h^3}{108}$. [2]

Using similar triangles,

$$\frac{r}{8} = \frac{h}{48} \quad \therefore r = \frac{h}{6}$$

$$\begin{aligned} \text{Volume of water, } V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left(\frac{h}{6}\right)^2 h \\ \therefore V &= \frac{\pi h^3}{108} \end{aligned}$$



- (ii) Find the rate of change of the water level when $h = 6$. [3]

$$\frac{dV}{dh} = \frac{\pi h^2}{36}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-5 = \frac{\pi h^2}{36} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-180}{\pi h^2}$$

$$\begin{aligned} \text{At } h = 6, \quad \frac{dh}{dt} &= \frac{-180}{\pi 6^2} \\ &\approx -1.59 \text{ cm}^3/\text{min} \quad (3\text{sf}) \end{aligned}$$

- (iii) State, with a reason, whether this rate will increase or decrease as t increases. [1]

As t increases, h decreases. Since $\frac{dh}{dt} = \frac{-180}{\pi h^2}$, $\frac{dh}{dt}$ is inversely proportional to h^2 , hence rate of change of water level increases when h decreases.

- 2 The displacement, y mm, of a mass fixed on a vertical spring can be described by the simple harmonic motion equation, $y = A \sin(\omega t)$, where A and ω are constants and t is the time in seconds after the mass is displaced from its equilibrium position, 0 mm.

Given that the maximum displacement of the mass is 20 mm and that the mass first returns to its equilibrium position after 0.25 seconds.

- (i) State the positive value of A .

[1]

$$A = 20$$

- (ii) Show that the value of ω is 4π radians per second.

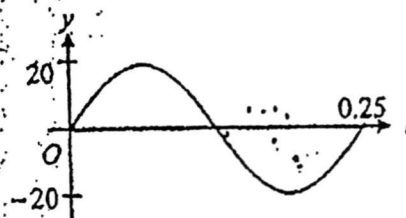
[2]

$$0 = 20 \sin \omega(0.25)$$

$$\sin \frac{1}{4} \omega = 0$$

$$\frac{1}{4} \omega = 0, \pi$$

$$\omega = 0 \text{ (rej)}, 4\pi$$



- (iii) Find the exact value of t when the mass first reach a position 10 mm below its equilibrium position.

[3]

$$-10 = 20 \sin 4\pi t$$

$$\sin 4\pi t = -\frac{1}{2}$$

$$\alpha = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$4\pi t = \pi + \frac{\pi}{6}$$

$$t = \frac{7\pi}{6} \times \frac{1}{4\pi}$$

$$= \frac{7}{24} \text{ s}$$

- 3 (i) Given that $f(x) = 2x^3 + ax^2 + bx - 30$ has a factor $(x+3)$ and leaves a remainder of -28 when divided by $(x-1)$. Find the values of a and of b and solve $f(x) = 0$. [6]

$$\begin{aligned} f(-3) &= 2(-3)^3 + a(-3)^2 + b(-3) - 30 = 0 \\ -54 + 9a - 3b - 30 &= 0 \\ 3a - b &= 28 \quad \dots(1) \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 + a(1) + b - 30 = -28 \\ a + b &= 0 \quad \dots(2) \end{aligned}$$

$$(1) + (2): 4a = 28$$

$$a = 7$$

$$b = -7$$

$$f(x) = 0$$

$$2x^3 + 7x^2 - 7x - 30 = 0$$

$$(x+3)(2x^2 + x - 10) = 0$$

$$(x+3)(x-2)(2x+5) = 0$$

$$x = -3 \text{ or } x = 2 \text{ or } x = -2.5$$

$$\begin{array}{r} 2x^2 + x - 10 \\ x+3 \overline{) 2x^3 + 7x^2 - 7x - 30} \\ \underline{2x^3 + 6x^2} \\ x^2 - 7x \\ \underline{x^2 + 3x} \\ -10x - 30 \\ \underline{-10x - 30} \\ 0 \end{array}$$

- (ii) Hence solve $2(y+1)^3 + a(y+1)^2 + by + b - 30 = 0$. [2]

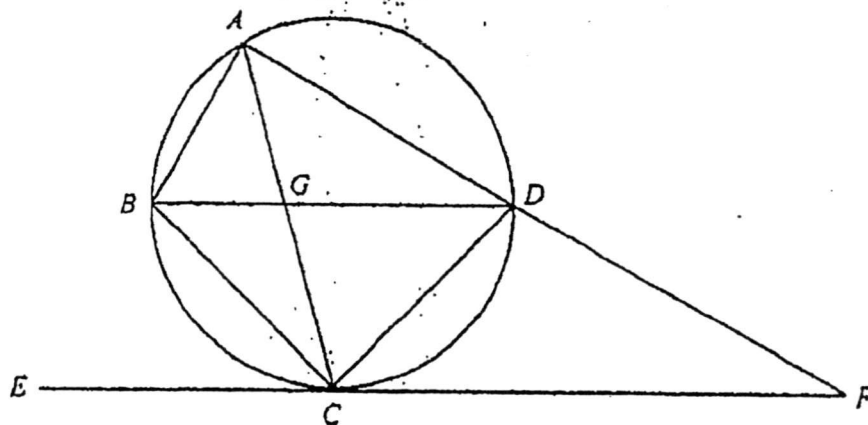
$$2(y+1)^3 + a(y+1)^2 + by + b - 30 = 0$$

$$2(y+1)^3 + a(y+1)^2 + b(y+1) - 30 = 0$$

$$\text{Let } x = y+1,$$

$$y+1 = -3 \text{ or } y+1 = 2 \text{ or } y+1 = -2.5$$

$$y = -4 \quad y = 1 \quad y = -3.5$$



The diagram shows points A, B, C and D lying on a circle. The chords BD and AC intersect at G. EF is a tangent to the circle at C. AD is produced meet the tangent at F and $\angle ABC = \angle BGC$.

Prove that

- (i) BD is parallel to EF,

[2]

$$\angle ACF = \angle ABC \quad (\angle s \text{ in alternate segment})$$

$$\angle ABC = \angle BGC \quad (\text{Given})$$

$$\therefore \angle BGC = \angle ACF$$

By the angle property of alternate angles, BD is parallel to EF.

- (ii) triangle CFD and triangle AFC are similar,

[2]

$$\angle CFD = \angle AFC \quad (\text{Common } \angle)$$

$$\angle DCF = \angle CAF \quad (\angle s \text{ in alternate segment})$$

Hence $\triangle CFD$ is similar to $\triangle AFC$.

- (iii) $FC^2 - FD^2 = FD \times DA$,

[3]

Since $\triangle CFD$ is similar to $\triangle AFC$,

$$\left. \begin{aligned} \frac{FD}{FC} &= \frac{CF}{AF} \\ FC^2 &= FD \times AF \end{aligned} \right\}$$

$$= FD \times (FD + DA)$$

$$= FD^2 + FD \times DA$$

$$\therefore FC^2 - FD^2 = FD \times DA \quad (\text{Proven})$$

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- 6 (ii) If the roots of the equation $gx^2 - hx - 1 = 0$ where g and h are constants, are α and β , find the value of g and of h . [2]

$$\alpha + \beta = \frac{h}{g}$$

$$\frac{1}{6} = \frac{h}{g}$$

$$\therefore h = \frac{g}{6} \dots\dots\dots(1)$$

$$\alpha\beta = \frac{-1}{g}$$

$$\frac{5}{16} = \frac{-1}{g}$$

$$g = \frac{-16}{5}$$

$$= -3\frac{1}{5}$$

$$\text{Sub } g = -3\frac{1}{5} \text{ into (1)}$$

$$\therefore h = -3\frac{1}{5} \div 6$$

$$= \frac{-8}{15}$$

Alternative solution:

$$x^2 - \frac{1}{6}x + \frac{5}{16} = 0$$

$$-\frac{16}{5}x^2 + \frac{8}{15}x - 1 = 0$$

$$gx^2 - hx - 1 = 0$$

$$\therefore g = \frac{-16}{5}$$

$$= -3\frac{1}{5}$$

$$\therefore -h = \frac{8}{15}$$

$$h = -\frac{8}{15}$$

- (b) Find the range of values of k for which $(k+3)x^2 + kx + 1$ is always positive for all real values of x . [4]

$$(k+3)x^2 + kx + 1 > 0$$

Since the expression is always positive,

$$k+3 > 0 \quad \text{and} \quad b^2 - 4ac < 0$$

$$k > -3 \quad k^2 - 4(k+3)(1) < 0$$

$$k^2 - 4k - 12 < 0$$

$$(k-6)(k+2) < 0$$

$$\therefore -2 < k < 6$$

Hence $k > -3$ and $-2 < k < 6$

$\therefore$ the solution is $-2 < k < 6$

7. Answer the whole of this question on a sheet of graph paper.

The table shows experimental values of two variables, x and y .

x	0.4	0.6	0.8	1.0	1.2
y	2.22	2.13	1.97	1.73	1.37

It is known that x and y are related by the equation $y^2 = (ax+1)x - b$, where a and b are constants.

- (i) On graph paper, plot $(y^2 - x)$ against x^2 , using a scale of 2 cm to represent 0.2 unit on the x^2 axis and 4 cm to represent 1 unit on the $(y^2 - x)$ axis. Draw a straight line graph to represent the equation $y^2 = (ax+1)x - b$. [3]

x^2	0.160	0.360	0.640	1.00	1.44
$y^2 - x$	4.53	3.94	3.08	1.99	0.677

- (ii) Use your graph to estimate the value of a and of b . [4]

$$y^2 = (ax+1)x - b$$

$$y^2 = ax^2 + x - b$$

$$y^2 - x = ax^2 - b$$

$$\text{Gradient} = a$$

$$\text{Gradient} = \frac{5-3.5}{0-0.5}$$

$$\therefore a = -3.00 \text{ (3sf)}$$

$$(y^2 - x) - \text{intercept} = -b$$

$$-b \approx 5$$

$$\therefore b = -5$$

- (iii) By drawing a suitable straight line on your graph, solve the equation $(a-2) = \frac{1+b}{x^2}$. [2]

$$(a-2) = \frac{1+b}{x^2}$$

$$ax^2 - 2x^2 = 1+b$$

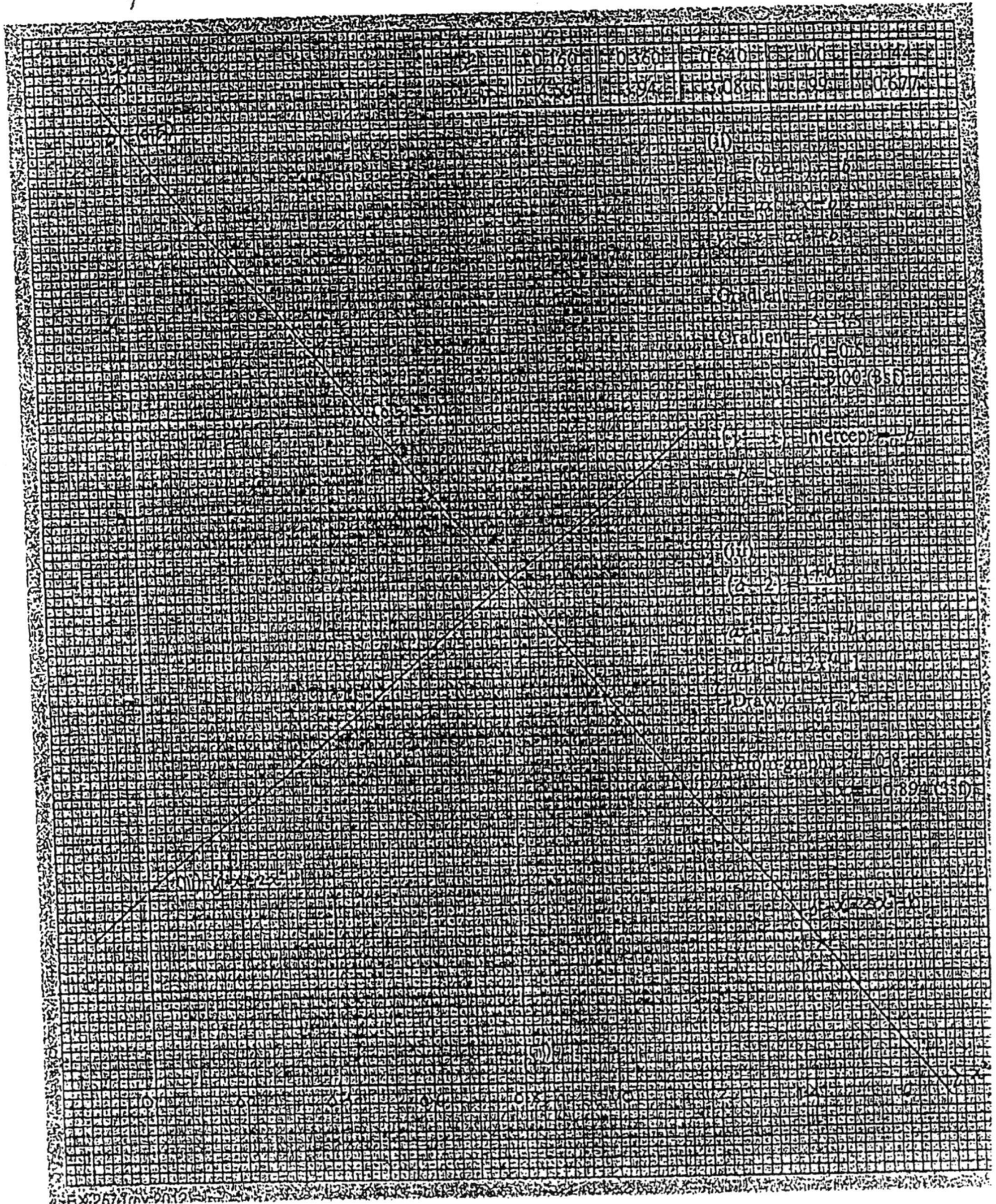
$$ax^2 - b = 2x^2 + 1$$

$$\text{Draw } y^2 - x = 2x^2 + 1$$

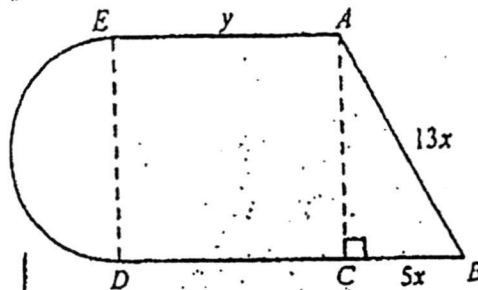
$$\text{From graph, } x^2 = 0.8$$

$$x = \pm 0.894 \text{ (3sf)}$$

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- 8 A piece of wire 160 cm long is bent to form the shape shown in the figure. This shape consists of a semi-circular arc whose diameter is given by the length DE , and a right-angled triangle ABC on the opposite ends of a rectangle of length y cm. The length of BC and AB are $5x$ cm and $13x$ cm respectively.



- (i) Express y in terms of x .

[2]

$$AC = \sqrt{(13x)^2 - (5x)^2} = 12x$$

Perimeter of figure = Length of wire

$$2y + \frac{\pi 12x}{2} + 13x + 5x = 160$$

$$2y + 6\pi x + 18x = 160$$

$$y + 3\pi x + 9x = 80$$

$$\therefore y = 80 - 3(\pi + 3)x$$

- (ii) Show that the area enclosed, $A \text{ cm}^2$, is given by $A = 960x - 6(3\pi + 13)x^2$.

[2]

$$A = \frac{1}{2}\pi(6x)^2 + y(12x) + \frac{1}{2}(5x)(12x)$$

$$= 18\pi x^2 + 30x^2 + 12x[80 - 3(\pi + 3)x]$$

$$= (18\pi + 30)x^2 + 960x - 36(\pi + 3)x^2$$

$$= [18\pi + 30 - 36(\pi + 3)]x^2 + 960x$$

$$= (18\pi + 30 - 36\pi - 108)x^2 + 960x$$

$$= (-18\pi - 78)x^2 + 960x$$

$$= 960x - 6(3\pi + 13)x^2$$

- 8 (iii) Determine the value of x for which A has a stationary value.

[3]

$$A = 960x - 6(3\pi + 13)x^2$$

$$\frac{dA}{dx} = 960 - 12(3\pi + 13)x$$

For stationary value of A ,

$$\frac{dA}{dx} = 0$$

$$960 - 12(3\pi + 13)x = 0$$

$$x = \frac{960}{12(3\pi + 13)}$$

$$\approx 3.567$$

$$= 3.57 \text{ (3 sf)}$$

- (iv) Find the stationary value of A and determine if it is a maximum or a minimum value.

[3]

$$\text{Stationary value of } A = 960(3.567) - 6(3\pi + 13)(3.567)^2$$

$$\approx 1712.39$$

$$= 1710 \text{ (3 sf)}$$

$$\frac{d^2A}{dx^2} = -12(3\pi + 13)$$

since $\frac{d^2A}{dx^2} < 0$, A is a maximum value.

- 9 (a) (i) Prove that $\cos A = \frac{\cos 2A}{\cos A} + \tan A \sin A$.

[3]

$$\begin{aligned} \text{RHS} &= \frac{\cos 2A}{\cos A} + \tan A \sin A \\ &= \frac{2\cos^2 A - 1}{\cos A} + \frac{\sin A}{\cos A} \cdot \sin A \\ &= \frac{2\cos^2 A - 1 + \sin^2 A}{\cos A} \\ &= \frac{2\cos^2 A - 1 + 1 - \cos^2 A}{\cos A} \\ &= \frac{\cos^2 A}{\cos A} \\ &= \cos A \\ &= \text{LHS} \end{aligned}$$

9. (a) (ii) Solve, for $0^\circ \leq A \leq 360^\circ$, $\cos A - \tan A \sin A = -1$.

[5]

$$\cos A - \tan A \sin A = -1$$

$$\frac{\cos 2A}{\cos A} = -1$$

$$2\cos^2 A - 1 = -\cos A$$

$$2\cos^2 A + \cos A - 1 = 0$$

$$(\cos A + 1)(2\cos A - 1) = 0$$

$$\cos A = -1$$

$$A = 180^\circ$$

$$\text{or } \cos A = \frac{1}{2}$$

$$A = 60^\circ$$

$$A = 60^\circ, 360^\circ - 60^\circ$$

$$= 60^\circ, 300^\circ$$

- (b) Given $\cos \theta = -\frac{4}{5}$ and θ is in the third quadrant. Without using a calculator, find the value of $\cos \frac{\theta}{2}$.

[3]

$$\cos \theta = -\frac{4}{5}$$

$$180^\circ < \theta < 270^\circ$$

$$2\cos^2 \frac{\theta}{2} - 1 = -\frac{4}{5}$$

$$90^\circ < \frac{\theta}{2} < 135^\circ$$

$$\cos^2 \frac{\theta}{2} = \frac{1}{10}$$

$$\cos \frac{\theta}{2} = \pm \frac{1}{\sqrt{10}}$$

$$= \pm \frac{\sqrt{10}}{10}$$

$$\text{Since } 90^\circ < \frac{\theta}{2} < 135^\circ, \cos \frac{\theta}{2} = -\frac{\sqrt{10}}{10}$$

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10 (a) Solve the equation $\log_2 \frac{1}{2} = \log_2 x - \log_4 (9x-2)$.

[3]

$$\log_2 \frac{1}{2} = \log_2 x - \log_4 (9x-2)$$

$$\log_4 (9x-2) = \log_2 x - \log_2 \frac{1}{2}$$

$$\frac{\log_2 (9x-2)}{\log_2 4} = \log_2 \left(x + \frac{1}{2} \right)$$

$$\frac{\log_2 (9x-2)}{2} = \log_2 2x$$

$$\log_2 (9x-2) = 2 \log_2 2x$$

$$\log_2 (9x-2) = \log_2 (2x)^2$$

$$\therefore 9x-2 = 4x^2$$

$$4x^2 - 9x + 2 = 0$$

$$(4x-1)(x-2) = 0$$

$$4x-1=0 \text{ or } x-2=0$$

$$x = \frac{1}{4} \text{ or } x = 2$$

(b) Given that $\log_2 (x+3) - (\log_2 y)(\log_2 2) = 2$, express y in terms of x .

[3]

$$\log_2 (x+3) - (\log_2 y)(\log_2 2) = 2$$

$$\log_2 (x+3) - \log_2 y \times \frac{1}{\log_2 8} = 2$$

$$\log_2 (x+3) - \frac{\log_2 y}{3} = 2$$

$$3 \log_2 (x+3) - \log_2 y = 6$$

$$\log_2 (x+3)^3 - \log_2 y = 6$$

$$\log_2 \frac{(x+3)^3}{y} = 6$$

$$\frac{(x+3)^3}{y} = 2^6$$

$$64y = (x+3)^3$$

$$y = \frac{(x+3)^3}{64}$$

$$\log_2 (x+3) - (\log_2 y)(\log_2 2) = 2$$

$$\log_2 (x+3) - \log_2 y \times \frac{\log_2 2}{\log_2 8} = 2$$

$$\log_2 (x+3) - \frac{\log_2 y}{3} = 2$$

$$\log_2 (x+3) - \log_2 \sqrt[3]{y} = 2$$

$$\log_2 \frac{(x+3)}{\sqrt[3]{y}} = 2$$

$$\frac{(x+3)}{\sqrt[3]{y}} = 2^2$$

$$\frac{(x+3)^3}{y} = 4^3$$

$$y = \frac{(x+3)^3}{64}$$

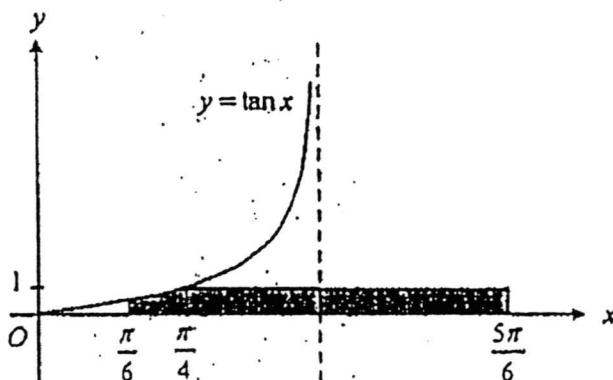
- 10 (c) (i) Differentiate
- $\ln \cos x$
- .

[1]

$$\begin{aligned}\frac{d}{dx} \ln \cos x &= \frac{-\sin x}{\cos x} \\ &= -\tan x\end{aligned}$$

- (ii) State the principal value of
- $\tan^{-1} 1$
- , giving your answer as a multiple of
- π
- . [1]

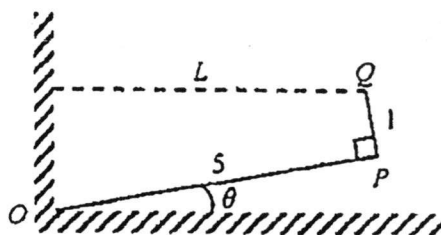
$$\text{Principal value of } \tan^{-1} 1 = \frac{\pi}{4}$$



The diagram shows part of the graph $y = \tan x$. The shaded region is bounded by the curve, the x axis, lines $x = \frac{5\pi}{6}$ and $y = 1$.

- (iii) Using your results from (i) and (ii), or otherwise, find the area of the shaded region. [4]

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \tan x \, dx + \left(\frac{5\pi}{6} - \frac{\pi}{4} \right) \\ &= -[\ln \cos x]_{\frac{\pi}{4}}^{\frac{5\pi}{6}} + \frac{7\pi}{12} \\ &= -\left[\ln \cos \frac{5\pi}{6} - \ln \cos \frac{\pi}{4} \right] + \frac{7\pi}{12} \\ &= -\ln \frac{1}{\sqrt{2}} + \ln \frac{\sqrt{3}}{2} + \frac{7\pi}{12} \\ &= \frac{1}{2} \ln \frac{3}{2} + \frac{7\pi}{12} \\ &= 2.04 \text{ units}^2 \text{ (3sf)}\end{aligned}$$



A L-shaped structure, OPQ , can be rotated about O . OP and PQ measures 5 m and 1 m respectively. OP makes an acute angle, θ , with the ground. Given that L m is the shortest distance from Q to the wall,

- (i) show that $L = 5 \cos \theta - \sin \theta$, [2]

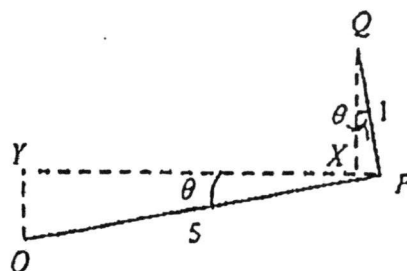
$$\cos \theta = \frac{PY}{5}$$

$$PY = 5 \cos \theta$$

$$\sin \theta = \frac{PX}{1}$$

$$PX = \sin \theta$$

$$\begin{aligned} \therefore L &= PY - PX \\ &= 5 \cos \theta - \sin \theta \end{aligned}$$



- (ii) express L in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, [4]

$$L = 5 \cos \theta - \sin \theta$$

$$= R \cos(\theta + \alpha)$$

$$= R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

$$R \cos \alpha = 5 \quad \dots (1) \quad R \sin \alpha = 1 \quad \dots (2)$$

$$(1)^2 + (2)^2: \quad R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 5^2 + 1^2$$

$$R = \sqrt{26}$$

$$= 5.10 \text{ (3sf)}$$

$$\frac{(2)}{(1)}: \quad \frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{5}$$

$$\tan \alpha = \frac{1}{5}$$

$$\alpha = \tan^{-1} \frac{1}{5}$$

$$= 11.31^\circ \text{ (2dp)}$$

$$\therefore L = 5.10 \cos(\theta + 11.3^\circ) \quad (3\text{sf}, 1\text{dp})$$

- 11 (iii) state the minimum value of L and find the corresponding value of θ , [3]

$$\text{Minimum } L = 0, \text{ when } \cos(\theta + 11.31^\circ) = 0$$

$$\theta + 11.31^\circ = 90^\circ$$

$$\theta = 78.7^\circ \text{ (1dp)}$$

- (iv) find the value of θ when $L = 3$, [2]

$$3 = \sqrt{26} \cos(\theta + 11.31^\circ)$$

$$\cos(\theta + 11.31^\circ) = \frac{3}{\sqrt{26}}$$

$$\alpha = \cos^{-1}\left(\frac{3}{\sqrt{26}}\right) = 53.96^\circ$$

$$\theta + 11.31^\circ = 53.96^\circ$$

$$\theta = 42.7^\circ \text{ (1dp)}$$

- (v) explain why the maximum value of L is not R . [1]

If $L = R$ then $\theta < 0^\circ$. Since $0^\circ \leq \theta < 90^\circ$, $\therefore$ maximum $L \neq R$.

[Since $\theta \geq 0^\circ$, maximum L occurs when $\theta = 0^\circ$, maximum $L = 5$.]

End of Paper

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